Unlocking the Secrets of Babylonian Numbers and Arithmetic: Exploring Ancient Mathematics

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Outline

- Babylonian Number System

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- Some History

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- Addition, Subtraction and Multiplication with Babylonian Numbers

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- Fractions and Division

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- Irrational Numbers

Babylonian Number System

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which represent 1 and 10, respectively.

The symbol representing 1 is called a wedge .

The symbol representing 10 is called a corner.

A Partial List of Babylonian Numbers

The following is a partial list of numbers from 1 to 59.

𒐕	𒐖	𒐗	𒐘	𒐙	𒐚	𒐛	𒐜	𒐝
1	2	3	4	5	6	7	8	9
𒌋	𒌋𒐕	𒌋𒐖	𒌋𒐚	𒎙	𒌍	𒐏	𒐐	𒐐𒐝
10	11	12	16	20	30	40	50	59

Additional Examples

We again emphasize that the Babylonian number system is positional and sexagesimal.

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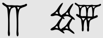
We provide more examples:

Decimal	Sexagesimal	Babylonian
$60 = 1 \cdot 60^1 + 0 \cdot 60^0$	$[1, 0]$	
$165 = 2 \cdot 60^1 + 45$	$[2, 45]$	
$3785 = 1 \cdot 60^2 + 3 \cdot 60 + 5$	$[1, 3, 5]$	

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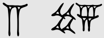
$165 = 2 \cdot 60^1 + 45$	$[2, 45]$	
$3785 = 1 \cdot 60^2 + 3 \cdot 60 + 5$	$[1, 3, 5]$	

As you can see from the table, we can express any positive integer using the sexagesimal number system.

Cuneiform Tablet

Cuneiform Tablet displayed in Library of Congress:

School exercise tables from 2200-1900 BCE.

Pythagorean Triples

Si.427 is a hand tablet from 1900-1600 BCE, created by an Babylonian surveyor. It's made out of clay, and the surveyor wrote on it with a stylus.

School exercise tables from 2200-1900 BCE. A land survey that maps out boundary lines, but the surveyor demonstrated a surprising level of knowledge by using what we today call "Pythagorean triples" to make precise right angles. If you paid attention in trigonometry class, you might remember this as the classic 3-4-5 triangle that creates mathematically perfect right angles.

Conversion from Decimal to Babylonian System

Using the Egyptian number method or the Russian peasant method, we can find the sexagesimal and the Babylonion representations of numbers. Let us work with number: 23342.

Russian Peasant Method

We can convert this calculation to the familiar table in the Russian peasant method:

	23342
2	23342
29	389
6	6

$$\Longrightarrow \boxed{6 \cdot 60^2 + 29 \cdot 60^1 + 2 \cdot 60^0 = 23342}$$

Russian Peasant Method

We can convert this calculation to the familiar table in the Russian peasant method:

$$\begin{array}{|r|}\hline \\ \hline\hline 2 \\ \hline 29 \\ \hline 6 \\ \hline \end{array} \quad \begin{array}{|r|}\hline 23342 \\ \hline\hline 23342 \\ \hline 389 \\ \hline 6 \\ \hline \end{array} \implies \boxed{6 \cdot 60^2 + 29 \cdot 60^1 + 2 \cdot 60^0 = 23342}$$

The last expression can be expressed in the sexagesimal number system as

$$6 \cdot 60^2 + 29 \cdot 60^1 + 2 \cdot 60^0 = 23342_{10} = [6, 29, 2].$$

Egyptian Method

We have

$$
\begin{array}{|r|}
\hline
\\
\hline\hline
2 \\
\hline
29 \\
\hline
6 \\
\hline
\end{array}
\quad
\begin{array}{|r|}
\hline
23342 \\
\hline\hline
1 \\
\hline
60 \\
\hline
3600 \\
\hline
\end{array}
\implies \boxed{6 \cdot 60^2 + 29 \cdot 60^1 + 2 \cdot 60^0 = 23342}
$$

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Babylonian Representation

The Babylonian form of the number 23342 is

$$𒐋\ 𒎙𒐎\ 𒈫 = [6, 29, 2].$$

Addition and Subtraction

Given

$$A = \text{𒌋𒐉 𒎙𒁹} \quad \text{and} \quad B = \text{𒎙𒐈 𒌍𒑄},$$

we compute the sum and the difference.

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Given

$$A = \text{𒌋𒐘 𒎙𒐕} \quad \text{and} \quad B = \text{𒎙𒐗 𒌋𒌋𒌋𒌋𒐜},$$

we compute the sum and the difference.

The sexagesimal forms of the numbers are

$$A = \text{𒌋𒐘 𒎙𒐕} = [14, 21] \quad \text{and} \quad B = \text{𒎙𒐗 𒌋𒌋𒌋𒌋𒐜} = [23, 48].$$

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We have

$$A + B = \text{𒌋𒐉 𒎙𒁹} + \text{𒎙𒐈 𒐏𒐍} = \text{𒌍𒐍 𒐎}$$

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$$\begin{aligned} A + B &= \text{𒌋𒐉 𒎙𒁹} + \text{𒎙𒐈 𒐏𒐍} = \text{𒌍𒐍 𒐎} \\ &= [14, 21] + [23, 48] = [38, 9]. \end{aligned}$$

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and

$B - A =$ 𒎙𒐈 𒐏𒐍 $-$ 𒌋𒐉 𒎙𒁹 $=$ 𒐎 𒎙𒐌

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$$= [23, 48] - [14, 21] = [9, 27].$$

Multiplication

Given

A = 𒌋𒁹 𒌋𒐛,   B = 𒐗 𒐘.

we compute the product.

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$$A = \text{𒌋𒁹 𒌋𒐊}, \quad B = \text{𒐈 𒐉}.$$

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$$A = \text{𒌋𒁹 𒌋𒐊} = [11, 15] \quad \text{and} \quad B = \text{𒐈 𒐉} = [3, 4].$$

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We have

𒌋𒁹 𒌋𒐊	$[11, 15]$
× 𒐈 𒐉	$\times\ [3, 4]$
𒐏𒐊	$[45, 0]$
+ 𒌍𒐈 𒐏𒐊	$+\ [33, 45, 0]$
𒌍𒐉 𒌍	$[34, 30, 0]$

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𒌋𒁹 𒌋𒐙	$[11, 15]$
× 𒐈 𒐉	$\times\ [3, 4]$
𒐏𒐙	$[45, 0]$
+ 𒌍𒐈 𒐏𒐙	$+\ [33, 45, 0]$
𒌍𒐉 𒌍	$[34, 30, 0]$

We performed the multiplication:

$$675 \times 184 = 124200$$

Representation of Fractions

We have

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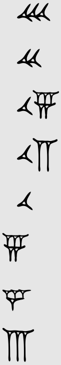

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$\frac{1}{8} = \frac{1}{2}\frac{1}{4} = [0;30][0;15] = [0;7,30] =$ 

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$$\frac{1}{8} = \frac{1}{2}\frac{1}{4} = [0;30][0;15] = [0;7,30] =$$ 

Division

Since the multiplicative inverse of 15 is $[0;4]$,

$$\frac{21}{15} = [21;0] \cdot [0;4]$$

We calculate the product

Sexagesimal	Explanation
$[21;0]$	
$[0;4]$	
$[0;1,24]$	$4 \cdot 21 = 84 = 1 \cdot 60 + 24$

Thus

$$\frac{21}{15} = [0;1,24].$$

More Division

We have

$\frac{1}{7}$	Integer Part	Difference
$\frac{60}{7} = 8 + \frac{4}{7}$	$b_1 = 8$	$\frac{4}{7}$
$\frac{240}{7} = 34 + \frac{2}{7}$	$b_2 = 34$	$\frac{2}{7}$
$\frac{120}{7} = 17 + \frac{1}{7}$	$b_3 = 17$	$\frac{1}{7}$
$\frac{60}{7} = 8 + \frac{4}{7}$	$b_4 = 8$	$\frac{4}{7}$

Thus

$$\frac{1}{7} = [0; 8, 34, 17, 8, 34, 17, 8, \ldots] = [0; \overline{8, 34, 17}]$$

.

A Table of Multiplicative Inverses 1

Based on the division process, having a table of multiplicative inverse of a set of numbers will be useful.

Babylonians created a table of inverses of numbers from 2 to 81 in sexagesimal number system. We give the inverses of numbers from 2 to 85:

$1/2 = [0;30]$	$1/16 = [0;3,45]$	$1/30 = [0;2]$
$1/3 = [0;20]$	$1/17 \approx [0;3,31,45,52]$	$1/31 \approx [0;1,56,7,44]$
$1/4 = [0;15]$	$1/18 = [0;3,20]$	$1/32 = [0;1,52,30]$
$1/5 = [0;12]$	$1/19 \approx [0;3,9,28,25]$	$1/33 \approx [0;1,49,5,27]$
$1/6 = [0;10]$	$1/20 = [0;3]$	$1/34 \approx [0;1,45,52,56]$
$1/7 \approx [0;8,34,17]$	$1/21 \approx [0;2,51,25,42]$	$1/35 \approx [0;1,42,51,25]$
$1/8 \approx [0;7,30]$	$1/22 \approx [0;2,43,38,10]$	$1/36 = [0;40]$
$1/9 = [0;6,40]$	$1/23 \approx [0;2,36,31,18]$	$1/37 \approx [0;1,37,17,50]$
$1/10 = [0;6]$	$1/24 = [0;2,30]$	$1/38 \approx [0;1,34,44,12]$
$1/11 \approx [0;5,27,16,21]$	$1/25 = [0;2,24]$	$1/39 \approx [0;1,32,18,27]$
$1/12 = [0;5]$	$1/26 \approx [0;2,18,27,41]$	$1/40 = [0;1,30]$
$1/13 \approx [0;4,36,55,23]$	$1/27 = [0;2,13,20]$	$1/41 \approx [0;1,27,48,17]$
$1/14 \approx [0;4,17,8,34]$	$1/28 \approx [0;2,8,34,17]$	$1/42 \approx [0;1,25,42,51]$
$1/15 = [0;4]$	$1/29 \approx [0;2,4,8,16]$	$1/43 \approx [0;1,23,43,15]$

A Table of Multiplicative Inverses 2

$1/44 \approx [0;1,21,49,5]$	$1/58 \approx [0;1,2,4,8]$	$1/72 = [0;0,50]$
$1/45 = [0;1,20]$	$1/59 \approx [0;1,1,1,1]$	$1/73 \approx [0;0,49,18,54]$
$1/46 \approx [0;1,18,15,39]$	$1/60 = [0;1]$	$1/74 \approx [0;0,48,38,55]$
$1/47 \approx [0;1,16,35,44]$	$1/61 \approx [0;0,59,0,59]$	$1/75 = [0;0,48]$
$1/48 = [0;1,15]$	$1/62 \approx [0;0,58,3,52]$	$1/76 \approx [0;0,47,22,6]$
$1/49 \approx [0;1,13,28,9]$	$1/63 \approx [0;0,57,8,34]$	$1/77 \approx [0;0,46,45,11]$
$1/50 = [0;1,12]$	$1/64 = [0;0,56,15]$	$1/78 \approx [0;0,46,9,13]$
$1/51 \approx [0;1,10,35,17]$	$1/65 \approx [0;0,55,23,4]$	$1/79 \approx [0;0,45,34,10]$
$1/52 \approx [0;1,19,13,50]$	$1/66 \approx [0;0,54,32,43]$	$1/80 = [0;0,45]$
$1/53 \approx [0;1,7,55,28]$	$1/67 \approx [0;0,53,43,52]$	$1/81 = [0;0,44,26,40]]$
$1/54 = [0;1,6,40]$	$1/68 \approx [0;0,52,56.28]$	$1/82 \approx [0;0,43,54,8]$
$1/55 \approx [0;1,5,27,16]$	$1/69 \approx [0;0,52,10,26]$	$1/83 \approx [0;0,43,22,24]$
$1/56 \approx [0;1,4,17,8]$	$1/70 \approx [0;0,51,25,42]$	$1/84 \approx [0;0,42,51,25]$
$1/57 \approx [0;1,3,9,28]$	$1/71 \approx [0;0,50,42,15]$	$1/85 \approx [0;0,42,21,10]$

Irrational Numbers

Based on the Babylonian algorithm

$$\sqrt{37} < 7 \implies 37 < 7\sqrt{37} \implies \frac{37}{7} < \sqrt{37}$$

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Thus

$$\frac{37}{7} < \sqrt{37} < 7.$$

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$$\sqrt{37} < 7 \implies 37 < 7\sqrt{37} \implies \frac{37}{7} < \sqrt{37}$$

Thus

$$\frac{37}{7} < \sqrt{37} < 7.$$

Thus, the first approximation R_1 is

$$\sqrt{37} \approx R_1 = \frac{1}{2} \cdot \frac{37}{7} + \frac{1}{2} \cdot 7 = \frac{43}{7}.$$

Irrational Numbers

Based on the Babylonian algorithm

$$\sqrt{37} < 7 \implies 37 < 7\sqrt{37} \implies \frac{37}{7} < \sqrt{37}$$

Thus

$$\frac{37}{7} < \sqrt{37} < 7.$$

Thus, the first approximation R_1 is

$$\sqrt{37} \approx R_1 = \frac{1}{2} \cdot \frac{37}{7} + \frac{1}{2} \cdot 7 = \frac{43}{7} \cdot$$

To find the second approximation, we apply the method again:

$$\sqrt{37} < \frac{43}{7} \implies 37 < \frac{43}{7}\sqrt{37} \implies \frac{37 \cdot 7}{43} < \sqrt{37}$$

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$$\frac{259}{43} < \sqrt{37} < \frac{43}{7}.$$

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To find the second approximation, we apply the method again:

$$\sqrt{37} < \frac{43}{7} \implies 37 < \frac{43}{7}\sqrt{37} \implies \frac{37 \cdot 7}{43} < \sqrt{37}$$

Thus

$$\frac{259}{43} < \sqrt{37} < \frac{43}{7}.$$

Thus, the second approximation R_2 is

$$\sqrt{37} \approx R_2 = \frac{1}{2}\left(\frac{43}{7} + \frac{259}{43}\right) = \frac{1831}{301}\cdot$$